MATHEMATICS

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AIM POINTS
MATHEMATICS
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XIth, XIIth, TARGET IIT-JEE (MAIN + ADVANCE) & COMPETITIVE EXAM. FOR XII (PQRS)

INVERSE TRIGONOMETRIC FUNCTIONS

& Their Properties

CONTENTS

THINGS TO REMEMBER

1. (i) $\sin^{-1}(\sin\theta) = \theta$, for all $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

   (ii) $\cos^{-1}(\cos\theta) = \theta$, for all $\theta \in [0, \pi]$

   (iii) $\tan^{-1}(\tan\theta) = \theta$, for all $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

   (iv) $\text{cosec}^{-1}(\text{cosec}\,\theta) = \theta$, for all $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, $\theta \neq 0$

   (v) $\sec^{-1}(\sec\theta) = \theta$, for all $\theta \in [0, \pi]$, $\theta \neq \frac{\pi}{2}$

   (vi) $\cot^{-1}(\cot\theta) = \theta$, for all $\theta \in (0, \pi)$

2. (i) $\sin(\sin^{-1}x) = x$, for all $x \in [-1, 1]$

   (ii) $\cos(\cos^{-1}x) = x$ for all $x \in [-1, 1]$

   (iii) $\tan(\tan^{-1}x) = x$ for all $x \in R$

   (iv) $\text{cosec}(\text{cosec}^{-1}x) = x$ for all $x \in (-\infty, -1] \cup [1, \infty)$

   (v) $\sec(\sec^{-1}x) = x$ for all $x \in (-\infty, -1] \cup [1, \infty)$

   (vi) $\cot(\cot^{-1}x) = x$ for all $x \in R$

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi-\theta, \text{if } \theta \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right] \\ \theta, \quad \text{if } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \pi-\theta, \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ -2\pi+\theta, \text{if } \theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \end{cases}$$

$$\cos^{-1}(\cos\theta) = \begin{cases} -\theta, \quad \text{if } \theta \in [-\pi, 0] \\ \theta, \quad \text{if } \theta \in [0, \pi] \\ 2\pi-\theta, \text{if } \theta \in \{\pi, 2\pi] \\ -2\pi+\theta, \text{if } \theta \in [2\pi, 3\pi] \end{cases}$$

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi-\theta, \text{if } \theta \in \left[-\frac{3\pi}{2}, -\frac{\pi}{2}\right] \\ \theta, \quad \text{if } \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \theta-\pi, \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ \theta-2\pi, \text{if } \theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \end{cases}$$

3. (i) $\sin^{-1}(-x) = -\sin^{-1}(x)$, for all $x \in [-1, 1]$
   (ii) $\cos^{-1}(-x) = \pi - \sin^{-1}(x)$, for all $x \in [-1, 1]$
   (iii) $\tan^{-1}(-x) = -\tan^{-1}(x)$, for all $x \in R$
   (iv) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}(x)$, for all $x \in (-\infty, -1] \cup [1, \infty)$
   (v) $\sec^{-1}(-x) = \pi - \sec^{-1}(x)$, for all $x \in (-\infty, -1] \cup [1, \infty)$
   (vi) $\cot^{-1}(-x) = \pi - \cot^{-1}(x)$, for all $x \in R$

4. (i) $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

   (ii) $\cos^{-1}\left(\frac{1}{x}\right) = \pi - \sec^{-1}x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

   (iii) $$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, & \text{for } x > 0 \\ -\pi + \cot^{-1}x, & \text{for } x < 0 \end{cases}$$

5. (i) $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$, for all $x \in [-1, 1]$

   (ii) $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}$, for all $x \in R$

   (iii) $\sec^{-1}x + \operatorname{csec}^{-1}x = \frac{\pi}{2}$, for all $x \in (-\infty, -1] \cup [1, \infty)$

6. (i) $\tan^{-1}x + \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } xy < 1 \\ \pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1}\left(\dfrac{x+y}{1-xy}\right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

6. (i) $\tan^{-1}x - \tan^{-1}y$

$$= \begin{cases} \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } xy < -1 \\ \pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1}\left(\dfrac{x-y}{1+xy}\right), & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

7. (i) $\sin^{-1}x + \sin^{-1}y$

$$= \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}, \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ \quad \text{or} \\ \quad \text{if } xy < 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}, \text{if } 0 < x, y \le 1 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\right\}, \text{if } -1 \le x, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases}$$

(ii) $\sin^{-1}x - \sin^{-1}y$

$$= \begin{cases} \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, \text{if } -1 \le x, y \le 1 \text{ and } x^2 + y^2 \le 1 \\ \quad \text{or} \\ \quad \text{if } xy > 0 \text{ and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, \text{if } 0 < x \le 1, -1 \le y \le 0 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\left\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\right\}, \text{if } -1 \le x < 0, 0 < y \le 1 \text{ and } x^2 + y^2 > 1 \end{cases}$$

8. (i) $\cos^{-1}x + \cos^{-1}y$

$$= \begin{cases} \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \text{if } -1 \le x, y \le 1 \text{ and } x + y \ge 0 \\ 2\pi - \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \text{if } -1 \le x, y \le 1 \text{ and } x + y \le 0 \end{cases}$$

(ii) $\cos^{-1}x - \cos^{-1}y$

$$= \begin{cases} \cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \text{if } -1 \le x, y \le 1 \text{ and } x \le y \\ -\cos^{-1}\left\{xy + \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \text{if } -1 \le y \le 0, 0 < x \le 1 \text{ and } x \ge y \end{cases}$$

9. (i) $$2\sin^{-1}x = \begin{cases} \sin^{-1}\left(2x\sqrt{1-x^2}\right) \quad, \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right) \quad, \text{if } -\frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right), \text{if } -1 \le x \le \frac{1}{2} \end{cases}$$

(ii) $3\sin^{-1}x = \begin{cases} \sin^{-1}(3x-4x^3) & ,\text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ \pi - \sin^{-1}(3x-4x^3) & ,\text{if } \frac{1}{2} < x \le 1 \\ -\pi - \sin^{-1}(3x-4x^3), & \text{if } -1 \le x < -\frac{1}{2} \end{cases}$

10. (i) $2\cos^{-1}x = \begin{cases} \cos^{-1}(2x^2-1) & ,\text{if } 0 \le x \le 1 \\ 2\pi - \cos^{-1}(2x^2-1), & \text{if } -1 \le x \le 0 \end{cases}$

(ii) $3\cos^{-1}x = \begin{cases} \cos^{-1}(4x^3-3x) & ,\text{if } \frac{1}{2} \le x \le 1 \\ 2\pi - \cos^{-1}(4x^3-3x), & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3-3x), & \text{if } -1 \le x \le \frac{1}{2} \end{cases}$

11. (i) $2\tan^{-1}x = \begin{cases} \tan^{-1}\left(\frac{2x}{1-x^2}\right) & ,\text{if } -1 < x < 1 \\ \pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) & ,\text{if } x > 1 \\ -\pi + \tan^{-1}\left(\frac{2x}{1-x^2}\right) & ,\text{if } x < -1 \end{cases}$

(ii) $3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } x > \frac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } x < -\frac{1}{\sqrt{3}} \end{cases}$

12. (i) $2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x < -1 \end{cases}$

(ii) $2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), \text{if } 0 \le x \le \infty \\ -\cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right), \text{if } -\infty < x \le 0 \end{cases}$

13. (i) $\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

$= \cot^{-1}\dfrac{\sqrt{1-x^2}}{x} = \sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right) = \operatorname{cosec}^{-1}\left(\dfrac{1}{x}\right)$

(ii) $\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)$

$= \cot^{-1}\left(\dfrac{x}{\sqrt{1-x^2}}\right) = \sec^{-1}\dfrac{1}{x} = \operatorname{cosec}^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

(iii) $\tan^{-1}x = \sin^{-1}\left(\dfrac{x}{\sqrt{1+x^2}}\right) = \cos^{-1}\left(\dfrac{1}{\sqrt{1+x^2}}\right)$

$= \cot^{-1}\left(\dfrac{1}{x}\right) = \sec^{-1}\sqrt{1+x^2} = \operatorname{cosec}^{-1}\left(\dfrac{\sqrt{1+x^2}}{x}\right)$

14. If $x_2, \ldots\ldots, x_n \in R$, then

$\tan^{-1}x_1 + \tan^{-1}x_2 + \ldots\ldots + \tan^{-1}x_n = \tan^{-1}\left(\dfrac{S_1 - S_3 + S_5 - S_7 + \ldots}{1 - S_2 + S_4 - S_6 + \ldots}\right)$

where S_k = Sum of the products of $x_1, x_2, \ldots, x_n$ takenk at a time.

EXERCISE-1

1. Prove that :

(i) $\sin^{-1}(\sin\theta) = \theta$, for all $\theta \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

(ii) $\cos^{-1}(\cos\theta) = \theta$, for all $\theta \in [0, \pi]$

(iii) $\tan^{-1}(\tan\theta) = \theta$, for all $\theta \in \left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

(iv) $\operatorname{cosec}^{-1}(\operatorname{cosec}\theta) = \theta$, for all $\theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right], \theta \neq 0$

(v) $\sec^{-1}(\sec\theta) = \theta$, for all $\theta \in [0, \pi], \theta \neq \frac{\pi}{2}$

(vi) $\cot^{-1}(\cot\theta) = \theta$, for all $\theta \in [0, \pi]$

2. Express each of the following in the simplest form :

(i) $\tan^{-1}\left\{\sqrt{\frac{1-\cos x}{1+\cos x}}\right\}, \pi - < x < \pi$

(ii) $\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$

(iii) $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right), -\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv) $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$

3. Write the following functions in the simplest form :

(i) $\tan^{-1}\left\{\frac{x}{\sqrt{a^2-x^2}}\right\}, -a < x < a$

(ii) $\tan^{-1}\left\{\frac{\sqrt{a-x}}{a+x}\right\}, -a < x < a$

(iii) $\sin^{-1}\left\{\frac{x}{\sqrt{x^2-a^2}}\right\}$

(iv) $\cos^{-1}\left\{\frac{x}{\sqrt{x^2-a^2}}\right\}$

4. Evaluate each of the following :

(i) $\sin\left(\sin^{-1}\frac{5}{13}\right)$

(ii) $\sin\left(\cos^{-1}\frac{4}{5}\right)$

(iii) $\sin\left(\tan^{-1}\frac{15}{8}\right)$

(iv) $\sin\left(\cot^{-1}\frac{4}{3}\right)$

(v) $\sin\left(\sec^{-1}\frac{17}{15}\right)$

(vi) $\sin\left(\operatorname{cosec}^{-1}\frac{17}{8}\right)$

5. Evaluate each of the following :

(i) $\cos\left(\cos^{-1}\frac{5}{13}\right)$

(ii) $\cos\left(\sin^{-1}\frac{8}{17}\right)$

(iii) $\cos\left(\tan^{-1}\frac{3}{4}\right)$

(iv) $\cos\left(\operatorname{cosec}^{-1}\frac{13}{12}\right)$

6. Evalueate :

$$\sin(\cot^{-1} x) = \sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right) = \frac{1}{\sqrt{1+x^2}}$$

7. Prove that :

(i) $\sin^{-1}(-x) = -\sin^{-1} x$, for all $x \in [-1, 1]$

(ii) $\cos^{-1}(-x) = \pi - \cos^{-1} x$, for all $x \in [-1, 1]$

(iii) $\tan^{-1}(-x) = -\tan^{-1} x$, for all $x \in R$

(iv) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1} x$, for all $x \in (-\infty, -1] \cup [1, \infty)$

(vi) $\cot^{-1}(-x) = \pi - \cot^{-1} x$, for all $x \in R$

8. Prove that :

$$\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x, \quad \text{for all } x \in (-\infty, -1] \cup [1, \infty)$$

9. Prove that :

$$\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2} \text{ for all } x \in (-\infty, -1] \cup [1, \infty)$$

10. Prove that :

(i) $\tan^{-1} x + \tan^{-1} y = \left(\frac{\sin x + \cos x}{\sqrt{2}}\right), -\frac{\pi}{4} < x < \frac{\pi}{4}$

(ii) $$\tan^{-1} x - \tan^{-1} y = \begin{cases} \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } xy > -1 \\ \pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x > 0, y < 0 \text{ and } xy > -1 \\ -\pi + \tan^{-1}\left(\frac{x-y}{1+xy}\right), & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

11. Prove that : $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$

12. Prove that : $\sin^{-1} \frac{8}{17} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{77}{85} = \tan^{-1}\left(\frac{77}{36}\right)$

13. Prove that : $\cos^{-1}\frac{12}{13} + \sin^{-1}\frac{3}{5} = \sin^{-1}\frac{56}{65}$

14. Prove that :

(i) $$\cos^{-1} x + \cos^{-1} y = \begin{cases} \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \text{if } -1 \le x, y \le 1 \text{ and } x + y \ge 0 \\ 2\pi - \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \text{if } -1 \le x, y \le 1 \text{ and } x + y \le 0 \end{cases}$$

(ii) $$\cos^{-1} x - \cos^{-1} y = \begin{cases} \cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \text{if } -1 \le x, y \le 1 \text{ and } x \le y \\ -\cos^{-1}\left\{xy - \sqrt{1-x^2}\sqrt{1-y^2}\right\}, \text{if } -1 \le x \le y0, 0 < 1 \text{ and } x \ge y \end{cases}$$

15. Prove that : $\cos^{-1}\frac{4}{5} + \cos^{-1}\frac{12}{13} = \cos^{-1}\frac{33}{65}$

16. Prove that : $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{8}{17} = \cos^{-1}\frac{84}{85}$

17. Prove that : $$2\sin^{-1}x = \begin{cases} \sin^{-1}\left(2x\sqrt{1-x^2}\right) & , \text{if } -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right) & , \text{if } -\frac{1}{\sqrt{2}} \le x \le 1 \\ -\pi - \sin^{-1}\left(2x\sqrt{1-x^2}\right) & , \text{if } -1 \le x \le \frac{1}{\sqrt{2}} \end{cases}$$

18. Prove that : $$3\tan^{-1}x = \begin{cases} \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) & , \text{if } -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) & , \text{if } x > \frac{1}{\sqrt{3}} \\ -\pi + \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) & , \text{if } x < \frac{1}{\sqrt{3}} \end{cases}$$

19. Prove that : $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

20. Evaluate : $\tan\left(2\tan^{-1}\frac{1}{5}\right)$

21. Prove that :

(i) $$2\tan^{-1}x = \begin{cases} \sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{if } -1 \le x \le 1 \\ \pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{if } x > 1 \\ -\pi - \sin^{-1}\left(\frac{2x}{1+x^2}\right), \text{if } x < -1 \end{cases}$$

(ii) $$2\tan^{-1}x = \begin{cases} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), \text{if } 0 \le x < \infty \\ -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), \text{if } -\infty < x \le 0 \end{cases}$$

22. Prove that : $\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2} + \cos^{-1}\frac{1-y^2}{1+y^2}\right] = \frac{x+y}{1-xy}$, if $|x| < 1$, $y > 0$ and $xy < 1$.

23. Prove that : $\tan^{-1}\sqrt{x} = \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right), x \in [0,1]$

24. Evaluate the following :
    (i) $\cos^{-1}(\cos 10)$
    (ii) $\tan^{-1}\{\tan(-6)\}$

25. Prove that :

(i) $$\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\} = \frac{\pi}{4}+\frac{x}{2}, 0 < x < \frac{\pi}{2}$$

(ii) $$\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\} = \frac{x}{2}, 0 < x < \frac{\pi}{2}$$

26. Prove that :

(i) $$\tan^{-1}\left\{\frac{\sqrt{1+\cos x}+\sqrt{1-\cos x}}{\sqrt{1+\cos x}-\sqrt{1-\cos x}}\right\} = \frac{\pi}{4}+\frac{x}{2}, \text{if } \pi < x < \frac{3\pi}{2}$$

(ii) $$\cot^{-1}\left\{\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right\} = \frac{\pi}{2}-\frac{x}{2}, \text{if } \frac{\pi}{2} < x < \pi$$

27. Prove that :

(i) $\tan^{-1}\left\{\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right\}=\dfrac{\pi}{4}-\dfrac{1}{2}\cos^{-1}x, 0<x<1$

(ii) $\tan^{-1}\left\{\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}=\dfrac{\pi}{4}+\dfrac{1}{2}\cos^{-1}x^2, -1<x<1$

28. Simplify each of the following :

(i) $\sin^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right), -\dfrac{\pi}{4}<x<\dfrac{\pi}{4}$

(ii) $\cos^{-1}\left(\dfrac{\sin x+\cos x}{\sqrt{2}}\right), \dfrac{\pi}{4}<x<\dfrac{5\pi}{4}$

29. Prove that : $\sin[\cot^{-1}\{\cos(\tan^{-1}x)\}]=\dfrac{\sqrt{x^2+1}}{x^2+2}$

30. If $x=\operatorname{cosec}\left[\tan^{-1}\left\{\cos\left(\cot^{-1}\left(\sec\left(\sin^{-1}a\right)\right)\right)\right\}\right]$

and $y=\sec\left[\cot^{-1}\left\{\sin\left(\tan^{-1}\left(\operatorname{cosec}\left(\cos^{-1}a\right)\right)\right)\right\}\right]$

where $a \in [0, 1]$. Find the relationship between x and y interms of a.

31. Simplify each of the following :

(i) $\tan^{-1}\left(\dfrac{a+bx}{b-ax}\right), x<\dfrac{b}{a}$

(ii) $\tan^{-1}\left(\dfrac{a\cos x-b\sin x}{b\cos x+a\sin x}\right), -\dfrac{\pi}{2}<x<\dfrac{\pi}{2}, \dfrac{a}{b}\tan x>-1$

(iii) $\tan^{-1}\left(\dfrac{3a^2x-x^3}{a^3-3ax^2}\right), -\dfrac{1}{\sqrt{3}}<\dfrac{x}{a}<\dfrac{1}{\sqrt{3}}$

32. Prove that :

(i) $\tan^{-1}\dfrac{1}{7}+\tan^{1}\dfrac{1}{13}=\tan^{1}\dfrac{2}{9}$

(ii) $\tan^{-1}\dfrac{1}{2}+\tan^{1}\dfrac{1}{5}=\tan^{1}\dfrac{1}{8}=\dfrac{\pi}{4}$

(iii) $\tan^{-1}\dfrac{3}{4}+\tan^{1}\dfrac{3}{5}=\tan^{1}\dfrac{8}{19}=\dfrac{\pi}{4}$

(ib) $\tan^{-1}\dfrac{1}{5}+\tan^{1}\dfrac{1}{7}=\tan^{1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{8}=\dfrac{\pi}{4}$

33. Prove that :

(i) $\sin^{-1}\frac{3}{5}+^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$

(ii) $\sin^{-1}\frac{4}{5}+^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$

(iii) $\sin^{-1}\frac{3}{5}+^{-1}\frac{8}{17}=\cos^{-1}\frac{36}{85}$

(iv) $\sin^{-1}\frac{3}{5}+\cos^{-1^{-1}}\frac{12}{13}=\cos^{-1}\frac{33}{65}=\sin^{-1}\frac{56}{65}$

34. Prove that : $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

35. If $a > b > c > 0$, prove that

$$\cot^{-1}\left(\frac{ab+1}{a-b}\right)+\cot^{-1}\left(\frac{bc+1}{b-c}\right)+\cot^{-1}\left(\frac{ca+1}{c-a}\right)=\pi$$

36. Prove that : $\cos^{-1}=2\sin^{-1}\sqrt{\frac{1-x}{2}}=2\cos^{-1}\sqrt{\frac{1+x}{2}}$

37. If $y=\cot^{-1}\left(\sqrt{\cos x}\right)-\tan^{-1}\left(\sqrt{\cos x}\right)$, prove that $\sin y=\tan^2\frac{x}{2}$.

38. Prove that $\tan^{-1}\frac{yz}{xr}+\tan^{-1}\frac{zx}{yr}+\tan^{-1}\frac{xy}{zr}=\frac{\pi}{2}$, where $x^2+y^2+z^2=r^2$.

39. (i) If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, prove that $x^2+y^2+z^2+2xyz=1$.

(ii) If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, prove that

(a) $x\sqrt{1-x^2}+y\sqrt{1-y^2}+z\sqrt{1-z^2}=2xyz$

(b) $x^4+y^4+z^4+2x^2y^2z^2=2(x^2y^2+y^2z^2+z^2x^2)$

40. Evaluate :

(i) $\tan\left\{2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right\}$

(ii) $\tan\left\{\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right\}$

41. If $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\}=\alpha$, then prove that $x^2=\sin 2\alpha$

42. Prove that : $\cos^{-1}\left(\frac{\cos\alpha+\cos\beta}{1+\cos\alpha\cos\beta}\right)=2\tan^{-1}\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\right)$.

43. Evaluate $\cos(2\cos^{-1}x + \sin^{-1}x)$ at $x = \frac{1}{5}$, where $0 \le \cos^{-1}x \le \pi$ and $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$.

44. Prove that : $\frac{\alpha^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right) + \frac{\beta^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right) = (\alpha+\beta)(\alpha^2+\beta^2)$

45. Prove that : $\tan^{-1}\frac{1-x}{1+x} - \tan^{-1}\frac{1-y}{1+y} = \sin^{-1}\frac{y-x}{\sqrt{1+x^2}\sqrt{1+y^2}}$

46. Evaluate : $\tan^{-1}\left(\frac{3\sin 2\alpha}{5+3\cos 2\alpha}\right) + \tan^{-1}\left(\frac{1}{4}\tan\alpha\right)$, where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

47. Prove that : $\tan\left\{\frac{\pi}{4} + \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} + \tan\left\{\frac{\pi}{4} - \frac{1}{2}\cos^{-1}\frac{a}{b}\right\} = \frac{2b}{a}$

48. Prove that : $\tan^{-1}\left\{\frac{\cos 2\alpha \sec 2\beta + \cos 2\beta \sec 2\alpha}{2}\right\} = \tan^{-1}\{\tan^2(\alpha+\beta)\tan^2(\alpha-\beta)\} + \tan^{-1}1$

49. Prove that : $\tan^{-1}\left(\frac{1}{2}\tan 2A\right) + \tan^{-1}(\cot A) + \tan^{-1}(\cot^3 A) = \begin{cases} 0, \text{if } \frac{\pi}{4} < A < \frac{\pi}{2} \\ \pi, \text{if } 0 < A < \frac{\pi}{4} \end{cases}$

50. If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1}x\right) = 1$, then find the value of x.

51. Solve the following equations :

(i) $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$

(ii) $\tan^{-1}2x + \tan^{-1}3x = \frac{\pi}{4}$

(iii) $\tan \tan^{-1}\frac{x-1}{x+1} + \tan^{-1}\frac{2x-1}{2x+1} = \tan^{-1}\frac{23}{36}$

52. Solve the following equations :

(i) $\sin^{-1}\frac{3x}{5} + \sin^{-1}\frac{4x}{5} = \sin^{-1}x$

(ii) $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

(iii) $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$

53. Evaluate the following :

(i) $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

(ii) $\cos^{-1}\left\{\cos\left(-\frac{\pi}{4}\right)\right\}$

(iii) $\tan^{-1}\left\{\tan\frac{3\pi}{4}\right\}$

(iv) $\tan^{-1}\left\{\tan\frac{2\pi}{3}\right\}$

(v) $\tan^{-1}\left\{\tan\frac{7\pi}{3}\right\}$

54. Evaluate the principal value : $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

55. Evaluate the following :

(i) $\cos\left(\sin^{-1}-\frac{3}{5}\right)$

(ii) $\tan\left(\cos^{-1}\frac{8}{17}\right)$

(iii) $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$

56. Prove the following results :

(i) $\cos^{-1}\left(\frac{5}{13}\right)=\tan^{-1}\left(\frac{12}{5}\right)$

(ii) $\sin^{-1}\left(-\frac{4}{5}\right)=\tan^{-1}\left(-\frac{4}{3}\right)=\cos^{-1}\left(-\frac{3}{5}\right)<\pi$

57. Prove the following results :

(i) $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$

(ii) $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}=\cos^{-1}\frac{3}{5}=\frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$

(iii) $\tan^{-1}\frac{2}{3}=\frac{1}{2}\tan^{-1}\frac{12}{5}$

(iv) $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$

(v) $2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$

(vi) $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{4}{7}$

58. Prove that :

(i) $\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{2}$

(ii) $\sin\left\{\tan^{-1}\frac{1-x^2}{2x}+\cos^{-1}\frac{1-x^2}{1+x^2}\right\}=1$

(iii) $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

59. Prove that : $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right)=\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)$

60. Solve the following equations for x :

(i) $\tan^{-1}2x+\tan^{-1}3x = n\pi+\frac{3\pi}{4}$

(ii) $\tan^{1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31}$

(iii) $\sin^{-1}x+\sin^{-1}2x=\frac{\pi}{3}$

(iv) $\cos^{-1}x+\sin^{-1}\frac{x}{2}=\frac{\pi}{6}$

(v) $\tan(\cos^{-1}x)=\sin\left(\cot^{-1}\frac{1}{2}\right)$

(vi) $\tan^{-1}\left(\frac{1-x}{1+x}\right)-\frac{1}{2}\tan^{-1}=0$, where $x>0$

(vii) $\tan^{-1}(x+2)+\tan^{-1}(x-2)=\tan^{-1}\left(\frac{8}{79}\right)$, $x>0$

(viii) $\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4}, 0<x<\sqrt{6}$

61. Write each of the following in the simplest form :

(i) $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}, x\neq 0$

(ii) $\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\}, -a < x < a$

(iii) $\sin^{-1}\left\{\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\}, -1 < x < 1$

(iv) $\cot^{-1}\frac{a}{\sqrt{x^2-a^2}}, |x| > a$

62. Find the value : $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\}$

63. Show that 2 $\tan^{-1}$ + x + $\sin^{-1}\frac{2x}{1+x^2}$ is constant for $x \geq 1$, find that constant.

64. Prove that : $\tan^{-1}\frac{2ab}{a^2-b^2}+\tan^{-1}\frac{2xy}{x^2-y^2}=\tan^{-1}\frac{2\alpha\beta}{\alpha^2-\beta^2}$

where $\alpha = ax - by$ and $\beta = ay + bx$.

65. For any a, b, x, y > 0, prove that :

$$\frac{2}{3}\tan^{-1}\left(\frac{3ab^2-a^3}{b^3-3a^2b}\right)+\frac{2}{3}\tan^{-1}\left(\frac{3xy^2-x^3}{y^3-3x^2y}\right)=\tan^{-1}\frac{2\alpha\beta}{\alpha^2-\beta^2}$$

where $\alpha = -ax + by$ and $\beta = bx + ay$.

EXERCISE-2

Answer each of the following questions in one word or one sentence or as per exact requirement of the questions :

1. Write the difference between maximum and minimum values of $\sin^{-1}x$ for $x \in [-1, 1]$.

2. If x > 1, then write the value of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ in terms of $\tan^{-1}x$.

3. If x < 0, then write the value of $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ in terms of $\tan^{-1}x$.

4. Write the value of $\tan^{-1}x + \tan\left(\frac{1}{x}\right)$ for x < 0.

5. If –1 < x < 0, then write the value of $\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

6. Write the value of $\cos\left(2\sin^{-1}\frac{1}{3}\right)$.

7. Write the value of $\cos^{-1}\left(\tan\frac{3\pi}{4}\right)$

8. Write the value of $\cos^2\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right)$

9. Write the value of $\cos^{-1}(\cos 6)$.

10. Write the value of $\tan^{-1}\left\{\tan\left(\frac{15\pi}{4}\right)\right\}$

11. Write the value of $\tan^{1}\frac{a}{b} - \tan^{-1}\left(\frac{a-b}{a+b}\right)$

12. Evaluate : $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$

13. If $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}x = \frac{\pi}{2}$, then find x.

14. Write the value of $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}x = \left(-\frac{1}{3}\right)$

15. What is the principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = ?$

EXERCISE-3

1. If $\alpha = \tan^{-1}\left(\tan\frac{5\pi}{4}\right)$ and $\beta = \tan^{-1}\left(-\tan\frac{2\pi}{3}\right)$, then

(a) $4\alpha = 3\beta$ (b) $4\alpha = 4\beta$ (c) $\alpha - \beta = \frac{7\pi}{12}$ (d) none of these

2. If $u = \cot^{-1}\sqrt{\tan\theta} - \tan^{1}\sqrt{\tan\theta}$ then, $\tan\left(\frac{\pi}{4} - \frac{u}{2}\right) =$

(a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$ (c) $-\pi$ (d) none of these

3. If $\alpha = \tan^{-1}\left(\frac{\sqrt{3}x}{2y-x}\right)$, $\beta = \left(\frac{2x-y}{\sqrt{3}y}\right)$, the $\pi - \beta =$

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{3}$

4. $\tan^{-1}\frac{1}{11} + \tan^{1}\frac{2}{11}$ is equal to

(a) 0 (b) $\frac{1}{2}$ (c) –1 (d) none of these

5. If $\tan^{-1} 3 + \tan^{-1}x = \tan^{-1} 8$, then x =

(a) 5 (b) $\frac{1}{5}$ (c) $\frac{5}{14}$ (d) $\frac{14}{5}$

6. The value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{3}$ (c) $\frac{10\pi}{3}$ (d) 0

7. If $4\cos^{-1} x + \sin^{-1} x = \pi$, then the value of x is

(a) $\frac{3}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{2}{\sqrt{3}}$

8. In a ΔABC, if C is a right angle, then

$$\tan \tan^{-1}\left(\frac{a}{b+c}\right)+\tan^{-1}\left(\frac{b}{c+a}\right)=$$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{5\pi}{2}$ (d) $\frac{\pi}{6}$

9. $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)=$

(a) 7 (b) 6 (c) 5 (d) none of these